Lecture 10

13.4 - Motion in Space.

Let $\vec{r}(t)$ be the position function of a particle.

The <u>velocity</u>, $\vec{v}(t)$ (note that velocity is a VECTOR), is

$$\boxed{\vec{v}(t) = \vec{r}\,'(t)}$$

and the <u>acceleration</u> (again a vector) is

$$\boxed{\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)}.$$

The <u>speed</u> of the particle is the rate of change in distance w.r.t. time, so

$$\boxed{\text{speed} = \frac{ds}{dt} = |\vec{r}\,'(t)| = |\vec{v}(t)|}$$

<u>Ex</u>: A particle has acceleration function

$$\vec{a}(t) = 4t\,\hat{i} + 6\sin(t)\,\hat{j} + e^t\,\hat{k}.$$

If its initial velocity is $\vec{v}(0) = 3\hat{j}$ and its initial position is $\vec{r}(0) = \vec{0}$, find its position function.

<u>Sol</u>: $\vec{v}(t) = \int \vec{a}(t)\,dt = (2t^2 + C_1)\hat{i} + (6\cos t + C_2)\hat{j} + (e^t + C_3)\hat{k}$

Now we use the initial velocity

$$\vec{v}(0) = (2(0)^2 + C_1)\hat{i} + (6\cos 0 + C_2)\hat{j} + (e^0 + C_3)\hat{k} = C_1\hat{i} + (6 + C_2)\hat{j} + (1 + C_3)\hat{k}$$

Now, the position

$$\vec{r}(t) = \left(\frac{2}{3}t^3 + D_1\right)\hat{i} + \left(6\sin t - 3t + D_2\right)\hat{j} + \left(e^t - t + D_3\right)\hat{k}$$

$$\vec{r}(0) = D_1\hat{i} + D_2\hat{j} + (1 + D_3)\hat{k}$$

and comparing this to the given initial position, we have

$$D_1 = D_2 = 0, \quad D_3 = -1$$

So, $\vec{r}(t) = \frac{2}{3}t^3\hat{i} + (6\sin t - 3t)\hat{j} + (e^t - t - 1)\hat{k}$

$\diamond$

If the particle has mass m, and acceleration $\vec{a}(t)$, the force it experiences is given by <u>Newton's Second Law of Motion</u> : $\vec{F}(t) = m\vec{a}(t)$

<u>Ex</u>: A projectile is fired with an angle of elevation α and initial velocity $\vec{v}_0$. Assuming all forces but gravity are negligible, find the position of particle at time t. What value of α maximizes the (horizontal) distance traveled?

Sol: We assume ground level is $y=0$ & we start at $x=0$. The only force that acts on the particle is gravity, so $\vec{F}(t) = -mg\hat{j}$, where $g \approx 9.8\, m/s^2$ is acceleration due to gravity. Since $\vec{F} = m\vec{a}$, we have $\vec{a}(t) = -g\hat{j}$. Then

$$\vec{v}(t) = \int \vec{a}(t)\,dt = -gt\hat{j} + \vec{C}.$$

$$\vec{v}(0) = \vec{C} \Rightarrow \vec{C} = \vec{v}_0, \text{ So } \vec{v}(t) = -gt\hat{j} + \vec{v}_0.$$

Now, we integrate for position:

$$\vec{r}(t) = \int \vec{v}(t)\,dt = -\frac{1}{2}gt^2\hat{j} + t\vec{v}_0 + \vec{D}$$

We assumed we started at the origin, so $\vec{r}(0) = \vec{0}$, from which we infer $\vec{D} = \vec{0}$. So $\vec{r}(t) = -\frac{1}{2}gt^2\hat{j} + t\vec{v}_0$.

Now, if we assume the initial speed is $|\vec{v}_0| = v_0$, then $\vec{v}_0 = v_0\cos\alpha\,\hat{i} + v_0\sin\alpha\,\hat{j}$. Thus

$$\vec{r}(t) = \underbrace{v_0 t\cos\alpha}_{= x}\,\hat{i} + \Big(\underbrace{v_0 t\sin\alpha - \tfrac{1}{2}gt^2}_{= y}\Big)\hat{j}$$

The distance traveled will be the value of x when $y=0$.

If $y=0$, then $y = t\left(v_0\sin\alpha - \frac{1}{2}gt\right) = 0 \Rightarrow t = 0, \frac{2v_0\sin\alpha}{g}$

The second value of t gives

$$d = x\left(\frac{2v_0 \sin\alpha}{g}\right) = \frac{v_0^2 \sin 2\alpha}{g}$$

which is at maximum when $\alpha = \frac{\pi}{4}$. $\diamond$

__Ex__: A projectile is fired with muzzle ~~velocity~~ speed $200\,m/s$ and angle of elevation $60°$. If the particle is fired from a distance of $10m$ off the ground, what is the distance covered by the projectile?

__Sol__: From the previous example, the equations of motion are

$$x = 200t\cos 60° = 100t$$

$$y = 200t\sin 60° - \frac{1}{2}gt^2 + \underbrace{10}_{\uparrow} = 100\sqrt{3}\,t - \frac{1}{2}gt^2 + 10$$

Since we are starting $10m$ off the ground, $\vec{r}(0) = 10\hat{j}$

The projectile hits the ground when $y = 0$, so

$$y = -\frac{1}{2}gt^2 + 100\sqrt{3}\,t + 10 = 0 \Rightarrow t = \frac{-100\sqrt{3} \pm \sqrt{30000 + 20g}}{-g}$$

We take the positive value of t $\quad (g \approx 9.8\,m/s^2)$

$$t = \frac{100\sqrt{3} + \sqrt{30000 + 20g}}{g} \approx 35.4$$

So, the distance traveled is

$$x = 100(t) \approx 3540 \text{ m}.$$

$\diamond$

Recall that the motion of a particle traveling along a path $\vec{r}(t)$ at some point P is best captured in the osculating plane of $\vec{r}(t)$ at that point. What we aim to do is to write the acceleration in terms of the tangent and normal vectors.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}}{v} \Rightarrow \vec{v} = v\vec{T} \quad (v = |\vec{v}|)$$

Take a derivative

$$\vec{v}\,' = \vec{a} = v'\vec{T} + v\vec{T}\,'. \quad \text{Now } \kappa = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{|\vec{T}\,'|}{v}$$

so $|\vec{T}\,'| = \kappa v$. Since $\vec{N} = \dfrac{\vec{T}\,'}{|\vec{T}\,'|}$,

We get $\vec{T}\,' = |\vec{T}\,'|\vec{N} = \kappa v \vec{N}$. Thus

$$\vec{a} = \underbrace{v'}_{\substack{a_T \\ \text{"tangential component"}}}\vec{T} + \underbrace{\kappa v^2}_{\substack{a_N \\ \text{"normal component"}}}\vec{N}$$

We can also find that

$$\boxed{a_T = \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{|\vec{r}\,'(t)|} \quad \& \quad a_N = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|}}$$

Ex: Find the normal and tangential components of acceleration for a particle moving along the trajectory

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$$

Sol: $\vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

$\vec{r}\,''(t) = -\cos t\,\hat{i} - \sin t\,\hat{j}$

$\vec{r}\,'(t)\cdot\vec{r}\,''(t) = \sin t\cos t - \cos t\sin t + 0 = 0$

So $a_T = 0$.

$$\vec{r}\,'(t)\times\vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \sin t\,\hat{i} - \cos t\,\hat{j} + \hat{k}$$

$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$|\vec{r}\,'(t)\times\vec{r}\,''(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

So, $a_N = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$\diamond$

Something convenient to note:

$\vec{a} = a_T \vec{T} + a_N \vec{N}$. Since $\vec{T} \cdot \vec{T} = \vec{N} \cdot \vec{N} = 1$

and $\vec{T} \cdot \vec{N} = 0$, we have

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a} = (a_T \vec{T} + a_N \vec{N}) \cdot (a_T \vec{T} + a_N \vec{N}) = a_T^{\,2} + a_N^{\,2}$$

Thus $|\vec{a}| = \sqrt{a_T^{\,2} + a_N^{\,2}}$.